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Higher-order energy-preserving difference scheme for the fourth-order nonlinear strain wave equation

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ABSTRACT

This paper focus on construction of high-order energy-preserving difference scheme for the fourth-order nonlinear strain wave equation with an energy conservation law. This target model is firstly transformed into an equivalent system by using the method of trigonometric scalar auxiliary variables. The resulting equivalent system possess a modified energy conservation law, and a fourth-order difference scheme with analogously discrete energy conservation law is developed based on the resulting equivalent system. The boundedness and convergence of the numerical solutions in the maximum norm are shown. The effectiveness of the difference scheme is verified by several numerical experiments.

1. Introduction

Consider the numerical method for system of the nonlinear strain fourth-order wave equation as follows

$$u_{tt} + \gamma\Delta^2 u - \Delta u + u^3 = 0,\ 0 < t \le T, \tag{1.1}$$

subject to the initial conditions

$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x), \tag{1.2}$$

where Δ is the Laplacian operator, γ is positive constant, $\varphi(x)$ and $\psi(x)$ are given smooth functions on Ω, see e.g., [1,2]. When $\gamma = 0$, Eq. (1.1) reduces to the cubic Klein-Gordon equation

$$u_{tt} - \Delta u + u^3 = 0, \tag{1.3}$$

which describes many phenomena in physic, such as superconductors and relativistic quantum mechanics, see e.g., [3,4]. In the elastic-plastic microstructure model studied by An and Peirce [5], the model (1.1)-(1.2) describes the elastic-plastic rod motion proposed by Avila in [6], which is used to analyze the influence of source term on the dynamics, forward and reverse of the undamped problem. Liu and Xu [7] proved that the global existence of solutions for initial boundary value problems with energy initial conditions. Shen et al. [1] proved that the solution blows up with any positive initial energy in a finite time.

In terms of numerical calculation, it is generally recognized that the structure-preserving algorithms are superior to other traditional methods, due to the former can retaining some inherent properties of a given system. In terms of this model (1.1), it has the following energy conservation law,

$$\frac{d}{dt}(\|u_t\|^2 + \gamma\|\Delta u\|^2 + \|\nabla u\|^2 + \frac{1}{2}\|u\|_4^4) = 0, \tag{1.4}$$

which can be obtained by taking inner product of Eq. (1.1) with u_t. Some structure-preserving algorithms have been proposed including the finite difference method, finite element method and others. For example, Achouri [8] proposed a conservative difference scheme for the 2D nonlinear

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fourth-order wave equation. Kadri [2] designed a linear conservative finite difference scheme for a fourth-order nonlinear strain wave equation. But it is regrettable that these work has only accuracy of second order. Recently, Yang et al. [9] proposed trigonometric scalar auxiliary variable (T-SAV) approach which is a new structure-preserving technology, and it has been proved to be efficient in constructing numerical schemes for a large number of gradient flows, while overcomes most of the disadvantages of SAV approach, see e.g., [10–14]. Given that, our goal is to develop and analyze the structure preserving difference scheme with higher-order accuracy by combining with the T-SAV approach.

The organizational structure of this paper is as follows. In Section 2, we transform the model (1.1)-(1.2) into an equivalent system by introducing triangular scalar auxiliary variable. In Section 3, we apply the second-order central difference in time and fourth-order approximation in space to discretize the equivalent system, and obtain a linear conservative difference scheme. In Section 4, the boundedness and convergence of the difference solutions are shown. In Section 5, several numerical examples are given to verify the theoretical results. Finally, a brief summary is placed in Section 6.

2. Equivalent system based on T-SAV approach

In this section, we use the T-SAV approach to transform the original fourth-order strain wave equation (1.1) into a new system satisfying the quadratic energy-conservation law. The resulting equivalent system provides an new platform for developing high-order linear structure-preserving scheme.

Let $L^p(\Omega)$ be space of measurable functions defined on Ω. The inner product and norm are defined as

$$(u,v)=\int_\Omega uv\,dx,\ \|u\|_{L^p}=(\int_\Omega |u|^p dx)^{1/p},\ 1\le p<\infty.$$

According to the idea of T-SAV approach, we introduce an auxiliary variable as follows

$$r(t)=\sin(F(t))+\delta, \tag{2.1}$$

where $F(t)=\frac{1}{2}\int_\Omega u^4 dx$, δ is a large enough positive constant to avoid singularity in the denominator of Eq. (2.4) and reduce the impact on u.

Taking the derivative of Eq. (2.1), we have

$$\frac{dr}{dt}=\cos(F(t))F'(t)=2\cos(F(t))\int_\Omega u^3u_t dx. \tag{2.2}$$

Noticing that $\sin^2 F+\cos^2 F=1$, we obtain

$$\frac{1}{\sqrt{1-(r-\delta)^2}}\frac{dr}{dt}=2\int_\Omega u^3u_t dx. \tag{2.3}$$

That is,

$$\frac{d}{dt}(\arcsin(r-\delta))=2\int_\Omega u^3u_t dx=\frac{2r}{\sin(F(t))+\delta}\int_\Omega u^3u_t dx. \tag{2.4}$$

As a result, Eq. (1.1) can be rewritten equivalently as

$$u_{tt}+\gamma\Delta^2u-\Delta u+\frac{r}{\sin(F(t))+\delta}u^3=0, \tag{2.5}$$

$$\frac{d}{dt}\arcsin(r-\delta)=(\frac{2ru^3}{\sin(F(t))+\delta},u_t). \tag{2.6}$$

Theorem 2.1. *The equivalent system* (2.5)-(2.6) *has a modified energy conservation law, that is*

$$E(t)=E(0),\ 0\le t\le T, \tag{2.7}$$

where

$$E(t)=\|u_t\|^2+\gamma\|\Delta u\|^2+\|\nabla u\|^2+\arcsin(r-\delta). \tag{2.8}$$

Proof. Taking inner product of equation (2.5) with u_t, we have

$$(u_{tt},u_t)+(\gamma\Delta^2u,u_t)-(\Delta u,u_t)+(\frac{r}{\sin(F(t))+\delta}u^3,u_t)=0. \tag{2.9}$$

This together with (2.6) gives

$$\frac{1}{2}\frac{d}{dt}[\|u_t\|^2+\gamma\|\Delta u\|^2+\|\nabla u\|^2+\arcsin(r-\delta)]=0, \tag{2.10}$$

i.e.,

$$\frac{d}{dt}E(t)=0.$$

It means that this proof of Theorem 2.1 is completed. □

3. High-order structure-preserving scheme

In this section, we aim to develop a high-order structure preserving difference scheme based on the above equivalent system (2.5)-(2.6). But before we do that, let's introduce some necessary notations and lemmas.

3.1. Notations and lemmas

Considering the computational interval $[x_L, x_R]$ with periodic L, let the mesh size $h = L/M$. Denote the grid points as $\Omega_h = \{x_i | x_i = x_L + ih, i = 0, 1, \cdots, M-1\}$, and let $U_h = \{u | u = (u_i), x_i \in \Omega_h\}$ be the space of grid function defined on Ω_h, equipped with discrete inner product and norms defined as

$$(u, v) = h\sum_{i=1}^{M} u_i v_i, \ \ \|u\| = \sqrt{(u,u)}, \ \ \|u\|_\infty = \max_{1\le i\le M} |u_i|, \ \ \forall u, v \in U_h.$$

Also we define the following difference operators for simplicity:

$$u_i^{n+\frac{1}{2}} = \frac{u_i^{n+1} + u_i^n}{2}, \ \hat{u}_i{}^n = \frac{u_i^{n+1} + u_i^{n-1}}{2},$$
$$\delta_t^+ u_i^n = \frac{u_i^{n+1} - u_i^n}{\tau}, \ \delta_{\hat{t}} u_i^n = \frac{u_i^{n+1} - u_i^{n-1}}{2\tau}, \ \delta_t^2 u_i^n = \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\tau^2},$$

and

$$\delta_x^+ u_i^n = \frac{u_{i+1}^n - u_i^n}{h}, \ \delta_x^- u_i^n = \frac{u_i^n - u_{i-1}^n}{h}, \ \delta_{\hat{x}} u_i^n = \frac{u_{i+1}^n - u_{i-1}^n}{2h},$$

and

$$\delta_x^2 u_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}, \ \delta_x^4 u_i^n = \delta_x^2(\delta_x^2 u_i^n), \ \delta_{\hat{x}}^2 u_i^n = \frac{u_{i+2}^n - 2u_i^n + u_{i-2}^n}{4h^2}.$$

3.2. Derivation of structure-preserving difference scheme

Denote U_i^n and u_i^n be the exact solution and the numerical solution of the problem (1.1)-(1.2) at the point (x_i, t_n), respectively, denote R^n and r^n be the exact solution and the numerical solution of $r(t)$ at the point t_n, respectively

Let $v_i^n = \frac{\partial^4 u}{\partial x^4}(x_i, t_n)$. Using Taylor's expansion, we get

$$\delta_x^4 u_i^n = v_i^n + \frac{h^2}{6}(\delta_x^2 v_i^n - \frac{h^2}{12}\frac{\partial^4 v}{\partial x^4}(\zeta_i, t_n)) + \mathcal{O}(h^4) = (1 + \frac{h^2}{6}\delta_x^2)v_i^n + \mathcal{O}(h^4),$$

where $\zeta_i \in (x_{i-1}, x_{i+1})$, see [15,16]. It means that we have

$$\frac{\partial^4 u}{\partial x^4}(x_i, t_n) = \mathcal{A}^{-1}\delta_x^4 u_i^n + \mathcal{O}(h^4), \tag{3.1}$$

where the difference operator $\mathcal{A} = \mathcal{I} + \frac{h^2}{6}\delta_x^2$ and $\mathcal{I}$ is the identity operator.

It is easy to verify that this matrix A corresponding to the operator $\mathcal{A}$ is a symmetric positive definite one. So, we denote $H = A^{-1}$ for brevity.

Lemma 3.1. *([17]) If $u(x) \in C^6[x_{i-1}, x_{i+1}]$, it holds that*

$$-\frac{d^2 u}{dx^2}(x_i) = -\frac{4}{3}\delta_x^2 u_i + \frac{1}{3}\delta_{\hat{x}}^2 u_i + \mathcal{O}(h^4).$$

Applying the second-order central difference in time, and (3.1) and Lemma 3.1 in space to the equivalent systems (2.5)-(2.6) yields that the fully-discrete scheme as follows

$$\delta_t^2 u^n + \gamma H \delta_x^4 \hat{u}^n - \frac{4}{3}\delta_x^2 \hat{u}^n + \frac{1}{3}\delta_{\hat{x}}^2 \hat{u}^n + b(u^n) r^n = 0, \tag{3.2}$$

$$\delta_t^+ \arcsin(r^n - \delta) = (2b(u^n)r^n, \delta_{\hat{t}} u^n), \tag{3.3}$$

where $u^n = [u_1^n, u_2^n, \cdots, u_M^n]^T$ and

$$b(u^n) = \frac{(u^n)^3}{\sin(F^n) + \delta}.$$

In order to implement the above three-level scheme (3.2)-(3.3), we need u^0 and u^1. Obviously, u^0 is given by (1.2), but u^1 need be determined.

According to the Taylor expansion, we have

$$(U^0)^2 U^{\frac{1}{2}} = (U^{\frac{1}{2}} - \frac{\tau}{2}U_t^{\frac{1}{2}} + \frac{\tau^2}{8}U_{tt}^{\frac{1}{2}} + \mathcal{O}(\tau^3))^2 U^{\frac{1}{2}} = (U^{\frac{1}{2}})^3 + \mathcal{O}(\tau),$$

and

$$\begin{aligned} U^1 &= U^0 + \tau U_t^0 + \frac{\tau^2}{2}U_{tt}^0 + \mathcal{O}(\tau^3) \\ &= U^0 + \tau U_t^0 + \frac{\tau^2}{2}(U_{tt}^{\frac{1}{2}} - \frac{\tau}{2}U_{ttt}^{\frac{1}{2}} + \mathcal{O}(\tau^2)) + \mathcal{O}(\tau^3) \end{aligned}$$

$$
\begin{aligned}
&= U^0 + \tau U_t^0 + \frac{\tau^2}{2} U_{tt}^{\frac{1}{2}} + \mathcal{O}(\tau^3) \\
&= U^0 + \tau U_t^0 + \frac{1}{2}\tau^2(-\gamma\Delta^2 U^{\frac{1}{2}} + \Delta U^{\frac{1}{2}} - (U^{\frac{1}{2}})^3) + \mathcal{O}(\tau^3) \\
&= U^0 + \tau U_t^0 - \frac{1}{2}\tau^2(\gamma\Delta^2 U^{\frac{1}{2}} - \Delta U^{\frac{1}{2}} + (U^0)^2 U^{\frac{1}{2}}) + \mathcal{O}(\tau^3),
\end{aligned}
$$

where Eq. (1.1) has been used. It means that u^1 can be obtained from the following formula

$$
u^1 = u^0 + \tau\psi - \frac{1}{2}\tau^2(\gamma H\delta_x^4 u^{\frac{1}{2}} - \frac{4}{3}\delta_x^2 u^{\frac{1}{2}} + \frac{1}{3}\delta_{\bar{x}}^2 u^{\frac{1}{2}} - (u^0)^2 u^{\frac{1}{2}}).
$$

That is,

$$
\delta_t u^0 + \frac{\gamma\tau}{2} A^{-1}\delta_x^4 u^{\frac{1}{2}} - \frac{2\tau}{3}\delta_x^2 u^{\frac{1}{2}} + \frac{\tau}{6}\delta_{\bar{x}}^2 u^{\frac{1}{2}} + \frac{\tau}{2}(u^0)^2 u^{\frac{1}{2}} = \psi(x). \tag{3.4}
$$

Given the above, we obtain the following difference scheme for solving the model problem (1.1)-(1.2) with periodic boundary conditions:

$$
\delta_t^2 u^n + \gamma H\delta_x^4 \hat{u}^n - \frac{4}{3}\delta_x^2 \hat{u}^n + \frac{1}{3}\delta_{\bar{x}}^2 \hat{u}^n + b(u^n) r^n = 0, \tag{3.5}
$$

$$
\delta_t^+ \arcsin(r^n - \delta) = (2b(u^n) r^n, \delta_{\hat{t}} u^n), \tag{3.6}
$$

$$
\delta_t^+ u^0 + \frac{\gamma\tau}{2} A^{-1}\delta_x^4 u^{\frac{1}{2}} - \frac{2\tau}{3}\delta_x^2 u^{\frac{1}{2}} + \frac{\tau}{6}\delta_{\bar{x}}^2 u^{\frac{1}{2}} + \frac{\tau}{2}(u^0)^2 u^{\frac{1}{2}} = \psi(x), \tag{3.7}
$$

where $\delta_{\hat{t}} u^0 = \psi$, $r^0 = \sin(\frac{1}{2}\int_\Omega \varphi(x)^4 dx) + \delta$.

3.3. Discrete conservation law

This section is devoted to study the conservation property of the difference scheme (3.5)-(3.7). But before we do that, let's introduce some necessary lemmas which plays an important role.

Lemma 3.2. *([18]) For any two mesh functions $w, v \in U_h$, and denote $H = B^T B$ by the Cholesky decomposition of H defined in (3.1), then we have*

$$
(H\delta_x^4 w, v) = (B\delta_x^2 w, B\delta_x^2 v).
$$

Lemma 3.3. *([19]) For any grid function $u \in U_h$, we have*

$$
\|u\| \leqslant \frac{L}{\sqrt{6}}\|\delta_x^+ u\|.
$$

Lemma 3.4. *([20,17]) For any two mesh functions $u, v \in U_h$, we have*

$$
(\delta_x^- u, v) = -(u, \delta_x^+ v),\ (\delta_{\bar{x}} u, v) = -(u, \delta_{\bar{x}} v), \tag{3.8}
$$

and

$$
(\delta_x^2 u, u) = -\|\delta_x^+ u\|^2,\ (\delta_{\bar{x}}^2 u, u) = -\|\delta_{\bar{x}} u\|^2,\ \|\delta_{\bar{x}} u\|^2 \le \|\delta_x^+ u\|^2. \tag{3.9}
$$

Based on above lemmas, we can obtain the follows result.

Theorem 3.5. *The difference scheme (3.5)-(3.7) satisfy the energy conservation law as follows*

$$
E^{n+1} = E^n,\ n = 0, 1, \cdots, N-1,
$$

where

$$
E^{n+1} = \|\delta_t^+ u^n\|^2 + \frac{\gamma}{2}(\|B\delta_x^2 u^{n+1}\|^2 + \|B\delta_x^2 u^n\|^2) + \frac{2}{3}(\|\delta_x^+ u^{n+1}\|^2 + \|\delta_x^+ u^n\|^2) - \frac{1}{6}(\|\delta_{\bar{x}} u^{n+1}\|^2 + \|\delta_{\bar{x}} u^n\|^2) + \arcsin(r^{n+1} - \delta).
$$

Proof. Taking the inner product of equation (3.5) with $\delta_{\hat{t}} u^n$, we get

$$
(\delta_t^2 u^n, \delta_{\hat{t}} u^n) + (\gamma H\delta_x^4 \hat{u}^n, \delta_{\hat{t}} u^n) - \frac{4}{3}(\delta_x^2 \hat{u}^n, \delta_{\hat{t}} u^n) + \frac{1}{3}(\delta_{\bar{x}}^2 \hat{u}^n, \delta_{\hat{t}} u^n) + (b(u^n) r^n, \delta_{\hat{t}} u^n) = 0. \tag{3.10}
$$

Noticing that

$$
(\delta_t^2 u^n, \delta_{\hat{t}} u^n) = \frac{1}{2\tau^3}(u^{n+1} - 2u^n + u^{n-1}, u^{n+1} - u^{n-1}) = \frac{1}{2\tau}(\|\delta_t^+ u^n\|^2 - \|\delta_t^+ u^{n-1}\|^2), \tag{3.11}
$$

and

$$
(H\delta_x^4 \hat{u}^n, \delta_{\hat{t}} u^n) = \frac{1}{4\tau}(B\delta_x^2 u^{n+1} + B\delta_x^2 u^{n-1}, B\delta_x^2 u^{n+1} - B\delta_x^2 u^{n-1}) = \frac{1}{4\tau}(\|B\delta_x^2 u^{n+1}\|^2 - \|B\delta_x^2 u^{n-1}\|^2), \tag{3.12}
$$

where Lemma 3.2 has been used.

Similarly, using Lemma 3.4, we obtain

$$(\delta_x^2\hat{u}^n,\delta_{\hat{t}}u^n)=\frac{1}{4\tau}(\delta_x^2u^{n+1}+\delta_x^2u^{n-1},u^{n+1}-u^{n-1})=-\frac{1}{4\tau}(\|\delta_x^+u^{n+1}\|^2-\|\delta_x^+u^{n-1}\|^2), \tag{3.13}$$

and

$$(\delta_{\bar{x}}^2\hat{u}^n,\delta_{\hat{t}}u^n)=\frac{1}{4\tau}(\delta_{\bar{x}}^2u^{n+1}+\delta_{\bar{x}}^2u^{n-1},u^{n+1}-u^{n-1})=-\frac{1}{4\tau}(\|\delta_{\bar{x}}u^{n+1}\|^2-\|\delta_{\bar{x}}u^{n-1}\|^2). \tag{3.14}$$

Substituting Eqs. (3.11)-(3.14) into (3.10), and combining (3.6) yields that

$$E^{n+1}=E^n.$$

This proof is completed. □

4. Numerical analysis

In this section, we mainly study the boundedness and convergence of the numerical solution computed by the difference scheme (3.5)-(3.7).

4.1. Boundedness

To prove boundedness of numerical solution, we need to introduce the following important inequality.

Lemma 4.1 *(Discrete Sobolev's inequality). ([21]) For any discrete function $u\in U_h^0$, there exist two constants M_1 and M_2 such that*

$$\|u^n\|_\infty\le M_1\|\delta_x^+u^n\|+M_2\|u^n\|.$$

Theorem 4.2. *The solution of the difference scheme (3.5)-(3.7) is bounded, i.e.,*

$$\|u^n\|_\infty\le C,$$

where C is a positive constant independent of h and τ.

Proof. Noticing that (3.7), u^1 can be obtained by u^0. It means that

$$E^0=\|\delta_t^+u^0\|^2+\frac{\gamma}{2}(\|B\delta_x^2u^1\|^2+\|B\delta_x^2u^0\|^2)+\frac{2}{3}(\|\delta_x^+u^1\|^2+\|\delta_x^+u^0\|^2)-\frac{1}{6}(\|\delta_{\bar{x}}u^1\|^2+\|\delta_{\bar{x}}u^0\|^2)+\arcsin(r^1-\delta)$$

is only determined by u^0 and ψ due to $r^1=\sin\left(\tau\left(2b\left(u^0\right)r^0,\psi\right)+\arcsin\left(r^0-\delta\right)\right)+\delta$. Thus, there is a positive constant c_0 such that $c_0=E^0$.

From Theorem 3.5, we obtain that

$$E^{n+1}=\|\delta_t^+u^n\|^2+\frac{\gamma}{2}(\|B\delta_x^2u^{n+1}\|^2+\|B\delta_x^2u^n\|^2)+\frac{2}{3}(\|\delta_x^+u^{n+1}\|^2+\|\delta_x^+u^n\|^2)-\frac{1}{6}(\|\delta_{\bar{x}}u^{n+1}\|^2+\|\delta_{\bar{x}}u^n\|^2)+\arcsin(r^{n+1}-\delta)=c_0.$$

Applying Lemma 3.4, it follows from above equality that

$$\|\delta_t^+u^n\|^2+\frac{\gamma}{2}(\|B\delta_x^2u^{n+1}\|^2+\|B\delta_x^2u^n\|^2)+\frac{1}{2}(\|\delta_x^+u^{n+1}\|^2+\|\delta_x^+u^n\|^2)+\arcsin(r^{n+1}-\delta)\le c_0.$$

This together with

$$\frac{\|u^{n+1}\|-\|u^n\|}{\tau}\le\|\delta_t^+u^n\|,$$

and $\arcsin(r^{n+1}-\delta)\in[-\pi/2,\pi/2]$, we have

$$\frac{\|u^{n+1}\|-\|u^n\|}{\tau}\le\|\delta_t^+u^n\|\le c_1,\ \ \|\delta_x^+u^n\|\le 2c_1,$$

where $c_1=\sqrt{c_0+\frac{\pi}{2}}$. It means that,

$$\|u^n\|\le n\tau c_1+\|u^0\|\le c_1T+\|u^0\|\triangleq c_2.$$

Using Discrete Sobolev's inequality in Lemma 4.1, one gets

$$\|u^n\|_\infty\le C,$$

where $C=2M_1c_1+M_2c_2$ is a positive constant independent of h and τ. This proof is completed. □

4.2. Convergence

Now, we focus on convergence of solution of the difference scheme (3.5)-(3.7). But before we do that, let's introduce the following important inequality.

Lemma 4.3 *(Gronwall inequality). ([22]) Suppose that the discrete grid function* $\{w^n \mid n = 0, 1, \cdots, N = T/\tau\}$ *satisfies the following inequality*

$$w^n - w^{n-1} \le A\tau w^n + B\tau w^{n-1} + C_n\tau,$$

where A, B *and* C_n *are non-negative constants, then*

$$\max_{1\le n\le N} |w^n| \le \left(w^0 + \tau\sum_{k=1}^{N} C_k\right) e^{2(A+B)T},$$

where τ *is sufficiently small, such that* $(A+B)\tau \le \frac{N-1}{2N} < \frac{1}{2}(N>1)$.

Based on the above inequality, and denote $e^n = U^n - u^n$, we can prove the following results.

Theorem 4.4. *Assume that the solution of the problem* (1.1)-(1.2) *is smooth enough, then we have*

$$\begin{aligned}&\|e^1\|_\infty \le (c_3M_1\sqrt{2L} + c_3M_2\sqrt{TL})(\tau^3 + h^4),\\ &\|e^n\|_\infty \le (M_1\sqrt{\exp(4c_8T)(2c_3^2L + c_4^2TL)} + M_2\frac{L\sqrt{6\exp(4c_8T)(2c_3^2L + c_4^2TL)}}{6})(\tau^2 + h^4),\end{aligned} \tag{4.1}$$

when $\tau < \tau_0 := \frac{1}{4c_8}$*, where* c_3, c_4, c_8 *are all positive constants independent of* h *and* τ*.*

Proof. According to the derivation in Section 3.2, we can easily obtain the error system as follows

$$\delta_t^2 e^n + \gamma H\delta_x^4\hat{e}^n - \frac{4}{3}\delta_x^2\hat{e}^n + \frac{1}{3}\delta_{\bar{x}}^2\hat{e}^n + b(U^n)R^n - b(u^n)r^n = q^n, \tag{4.2}$$

$$\delta_t^+ e^0 + \frac{\gamma\tau}{2}H\delta_x^4 e^{\frac{1}{2}} - \frac{2\tau}{3}\delta_x^2 e^{\frac{1}{2}} + \frac{\tau}{6}\delta_{\bar{x}}^2 e^{\frac{1}{2}} + \frac{\tau}{2}(U^0)^2U^{\frac{1}{2}} - \frac{\tau}{2}(u^0)^2u^{\frac{1}{2}} = w, \tag{4.3}$$

where $q^n = [q_1^n, q_2^n, \cdots, q_M^n]^T$, $w = [w_1, w_2, \cdots, w_M]^T$ are the truncation errors. That is,

$$\begin{aligned}&\delta_t^2 U^n + \gamma H\delta_x^4\hat{U}^n - \frac{4}{3}\delta_x^2\hat{U}^n + \frac{1}{3}\delta_{\bar{x}}^2\hat{U}^n + b(U^n)R^n = q^n,\\ &\delta_t^+ U^0 + \frac{\gamma\tau}{2}H\delta_x^4 U^{\frac{1}{2}} - \frac{2\tau}{3}\delta_x^2 U^{\frac{1}{2}} + \frac{\tau}{6}\delta_{\bar{x}}^2 U^{\frac{1}{2}} + \frac{\tau}{2}\left(U^0\right)^2 U^{\frac{1}{2}} = w,\end{aligned}$$

thus there are two positive constants c_3, c_4 such that

$$|w_i| \le c_3(h^4 + \tau^3),\ |q_i^n| \le c_4(h^4 + \tau^2),\ 1 \le i \le M, 1 \le n \le N.$$

(I) Now we consider the convergence result when $n = 0$. At first, from (1.2), we get $\|e^0\| = 0$. Taking the inner product (4.3) with $\delta_t^+ e^0$, we obtain

$$(\delta_t^+ e^0, \delta_t^+ e^0) + \frac{\gamma\tau}{2}(H\delta_x^4 e^{\frac{1}{2}}, \delta_t^+ e^0) - \frac{2\tau}{3}(\delta_x^2 e^{\frac{1}{2}}, \delta_t^+ e^0) + \frac{\tau}{6}(\delta_x^{2o} e^{\frac{1}{2}}, \delta_t^+ e^0) + \frac{\tau}{2}((U^0)^2U^{\frac{1}{2}} - (u^0)^2u^{\frac{1}{2}}, \delta_t^+ e^0) = (w, \delta_t^+ e^0). \tag{4.4}$$

Applying Lemma 3.2, we have

$$\frac{\gamma\tau}{2}(H\delta_x^4 e^{\frac{1}{2}}, \delta_t^+ e^0) = \frac{\gamma}{4}(H\delta_x^4(e^1 + e^0), e^1 - e^0) = \frac{\gamma}{4}(B\delta_x^2 e^1, B\delta_x^2 e^1) = \frac{\gamma}{4}\|B\delta_x^2 e^1\|^2. \tag{4.5}$$

Applying Lemma 3.4, we get

$$\frac{2}{3}\tau(\delta_x^2 e^{\frac{1}{2}}, \delta_t^+ e^0) = \frac{1}{3}(\delta_x^2 e^1 + \delta_x^2 e^0, e^1 - e^0) = -\frac{1}{3}(\|\delta_x^+ e^1\|^2 - \|\delta_x^+ e^0\|^2) = -\frac{1}{3}\|\delta_x^+ e^1\|^2, \tag{4.6}$$

and

$$\frac{\tau}{6}(\delta_{\bar{x}}^2 e^{\frac{1}{2}}, \delta_t^+ e^0) = \frac{1}{12}(\delta_{\bar{x}}^2 e^1 + \delta_{\bar{x}}^2 e^0, e^1 - e^0) = -\frac{1}{12}(\|\delta_{\bar{x}} e^1\|^2 - \|\delta_{\bar{x}} e^0\|^2) = -\frac{1}{12}\|\delta_{\bar{x}} e^1\|^2. \tag{4.7}$$

Inserting (4.5)-(4.7) into (4.4) yields that

$$\|\delta_t^+ e^0\|^2 + \frac{\gamma}{4}\|B\delta_x^2 e^1\|^2 + \frac{1}{3}\|\delta_x^+ e^1\|^2 - \frac{1}{12}\|\delta_{\bar{x}} e^1\|^2 + \frac{1}{4}\|u^0 e^1\|^2 = (w, \delta_t^+ e^0). \tag{4.8}$$

Noticing that $\|\delta_x^+ e\| \le \|\delta_{\bar{x}} e\|$, thus we have

$$\frac{1}{4}\|\delta_x^+ e^1\|^2 \le \frac{1}{3}\|\delta_x^+ e^1\|^2 - \frac{1}{12}\|\delta_{\bar{x}} e^1\|^2.$$

This together with (4.8) gives that

$$\|\delta_t^+ e^0\|^2 + \frac{\gamma}{4}\|B\delta_x^2 e^1\|^2 + \frac{1}{4}\|\delta_x^+ e^1\|^2 + \frac{1}{4}\|u^0 e^1\|^2 \le (w, \delta_t^+ e^0).$$

Since

$$\frac{\|e^1\|^2}{\tau} \le \|\delta_t^+ e^0\|^2,\ (w, \delta_t^+ e^0) \le \frac{1}{2}(\|w\|^2 + \|\delta_t^+ e^0\|^2).$$

Thus, we have

$$\frac{1}{2\tau}\|e^1\|^2 + \frac{\gamma}{4}\|B\delta_x^2 e^1\|^2 + \frac{1}{4}\|\delta_x^+ e^1\|^2 + \frac{1}{4}\|u^0 e^1\|^2 \le \frac{1}{2}\|w\|^2.$$

It implies that

$$\|e^1\|^2 \le c_3^2 \tau L(\tau^3 + h^4)^2 \le c_3^2 T L(h^4 + \tau^3)^2, \|\delta_x^+ e^1\|^2 \le 2c_3^2 L(h^4 + \tau^3)^2.$$

Using Lemma 4.1, we have

$$\|e^1\|_\infty \le (c_3 M_1 \sqrt{2L} + c_3 M_2 \sqrt{TL})(h^4 + \tau^3). \tag{4.9}$$

(II) Now we consider the convergence result when $n \ge 1$.

Taking the inner product with both sides of (4.2) with $\delta_{\hat{t}} e^n$, we obtain

$$(\delta_t^2 e^n, \delta_{\hat{t}} e^n) + (\gamma H \delta_x^4 \hat{e}^n, \delta_{\hat{t}} e^n) - \frac{4}{3}(\delta_x^2 \hat{e}, \delta_{\hat{t}} e^n) + \frac{1}{3}(\delta_{\bar{x}}^2 \hat{e}^n, \delta_{\hat{t}} e^n) + (G, \delta_{\hat{t}} e^n) = (q^n, \delta_{\hat{t}} e^n), \tag{4.10}$$

where

$$G = b(U^n)R^n - b(u^n)r^n.$$

Noticing that

$$(\delta_t^2 e^n, \delta_{\hat{t}} e^n) = (\frac{e^{n+1} - 2e^n + e^{n-1}}{\tau^2}, \frac{e^{n+1} - e^{n-1}}{2\tau}) = \frac{1}{2\tau}(\|\delta_t^+ e^n\|^2 - \|\delta_t^+ e^{n-1}\|^2). \tag{4.11}$$

Applying Lemma 3.2, we have

$$(\gamma H \delta_x^4 \hat{e}^n, \delta_{\hat{t}} e^n) = (\gamma H \delta_x^4 \hat{e}^n, \frac{e^{n+1} - e^{n-1}}{2\tau}) = \frac{\gamma}{4\tau}(\|B\delta_x^2 e^{n+1}\|^2 - \|B\delta_x^2 e^{n-1}\|^2). \tag{4.12}$$

Similarly, using Lemma 3.4, we get

$$\frac{4}{3}(\delta_x^2 \hat{e}^n, \delta_{\hat{t}} e^n) = \frac{4}{3}(\frac{\delta_x^2 e^{n+1} + \delta_x^2 e^{n-1}}{2}, \frac{e^{n+1} - e^{n-1}}{2\tau}) = -\frac{1}{3\tau}(\|\delta_x^+ e^{n+1}\|^2 - \|\delta_x^+ e^{n-1}\|^2), \tag{4.13}$$

and

$$\frac{1}{3}(\delta_{\bar{x}}^2 \hat{e}^n, \delta_{\hat{t}} e^n) = \frac{1}{3}(\frac{\delta_{\bar{x}}^2 e^{n+1} + \delta_{\bar{x}}^2 e^{n-1}}{2}, \frac{e^{n+1} - e^{n-1}}{2\tau}) = -\frac{1}{12\tau}(\|\delta_{\bar{x}} e^{n+1}\|^2 - \|\delta_{\bar{x}} e^{n-1}\|^2). \tag{4.14}$$

For simplicity, denote

$$A^n = \|\delta_t^+ e^n\|^2 + \frac{\gamma}{2}(\|B\delta_x^2 e^{n+1}\|^2 + \|B\delta_x^2 e^n\|^2) + \frac{2}{3}(\|\delta_x^+ e^{n+1}\|^2 + \|\delta_x^+ e^n\|^2) - \frac{1}{6}(\|\delta_{\bar{x}} e^{n+1}\|^2 + \|\delta_{\bar{x}} e^n\|^2).$$

Substituting into (4.11)-(4.14) into (4.10), then we have

$$A^n - A^{n-1} = 2\tau((q^n, \delta_{\hat{t}} e^n) - (G, \delta_{\hat{t}} e^n)). \tag{4.15}$$

Since

$$f(U^n) - f(u^n) = f'(\xi^n)(U^n - u^n) \le c_5 |e^n|,$$
$$r^{n+1} = \sin((b(u^n)r^n, u^{n+1} - u^{n-1}) + \arcsin(r^n - \delta)) + \delta,$$

where $c_5 = \max |f'(\xi^n)|$ and ξ^n is on the segment that connects U^n and u^n.

It means that when δ is large enough, we have

$$\begin{aligned}\|G\|^2 &= \| \frac{R^n f(U^n)}{\sin(\int_\Omega \frac{1}{2}(U^n)^4 dx) + \delta} - \frac{r^n f(u^n)}{\sin(\int_\Omega \frac{1}{2}(u^n)^4 dx) + \delta} \|^2 \\ &\le c_6 \|f(U^n) - f(u^n)\|^2 \le c_7 \|e^n\|^2,\end{aligned}$$

where c_6 is a positive constant, $c_7 = c_5^2 c_6$. Also, we have

$$\|\frac{e^{n+1} - e^{n-1}}{2\tau}\|^2 \le \frac{1}{2}(\|\delta_t^+ e^n\|^2 + \|\delta_t^+ e^{n-1}\|^2).$$

As a result, it follows from Lemma 3.3 that

$$\begin{aligned}&(q^n, \delta_{\hat{t}} e^n) - (G, \delta_{\hat{t}} e^n) \\ &\le \|q^n\|\|\delta_{\hat{t}} e^n\| + \|G\|\|\delta_{\hat{t}} e^n\| \\ &\le \frac{1}{2}(\|q^n\|^2 + \|G\|^2) + \|\delta_{\hat{t}} e^n\|^2 \\ &\le \frac{1}{2}\|q^n\|^2 + \|\delta_{\hat{t}} e^n\|^2 + c_3\|e^n\|^2 \\ &\le \frac{1}{2}\|q^n\|^2 + \frac{1}{2}(\|\delta_t^+ e^n\|^2 + \|\delta_t^+ e^{n-1}\|^2) + c_3\|e^n\|^2 \\ &\le \frac{1}{2}\|q^n\|^2 + \frac{1}{2}(\|\delta_t^+ e^n\|^2 + \|\delta_t^+ e^{n-1}\|^2) + c_3\|\delta_x^+ e^n\|^2.\end{aligned} \tag{4.16}$$

It means that

$$A^n - A^{n-1} \le \tau\|q^n\|^2 + \tau(\|\delta_t^+ e^n\|^2 + \|\delta_t^+ e^{n-1}\|^2) + 2c_3\tau\|\delta_x^+ e^n\|^2.$$

Noticing that

$$A^n \ge \|\delta_t^+ e^n\|^2 + \frac{\gamma}{2}(\|B\delta_x^2 e^{n+1}\|^2 + \|B\delta_x^2 e^n\|^2) + \frac{1}{2}(\|\delta_x^+ e^{n+1}\|^2 + \|\delta_x^+ e^n\|^2),$$

where Lemma 3.4 has been used. Hence, we have

$$A^n - A^{n-1} \le c_8\tau A^n + c_8\tau A^{n-1} + \tau\|q^n\|^2, \tag{4.17}$$

where $c_8 = \max\{1, 2c_7\}$, when $\tau \le \frac{N-1}{4c_8 N} \le \tau_0 := \frac{1}{4c_8}$, using discrete Gronwall inequality, we obtain

$$A^n \le \exp(4c_8 T)(A^0 + \tau\sum_{n=1}^{N}\|q^n\|^2). \tag{4.18}$$

According to (4.8), we have

$$\frac{1}{2}\left\|\delta_t^+ e^0\right\|^2 + \frac{\gamma}{4}\left\|B\delta_x^2 e^1\right\|^2 + \frac{1}{4}\left\|\delta_x^+ e^1\right\|^2 \le \frac{1}{2}\|w\|^2.$$

Noticing that $e^0 = 0$, thus we have

$$\begin{aligned}
A^0 &= \|\delta_t^+ e^0\|^2 + \frac{\gamma}{2}(\|B\delta_x^2 e^1\|^2 + \|B\delta_x^2 e^0\|^2) + \frac{2}{3}(\|\delta_x^+ e^1\|^2 + \|\delta_x^+ e^0\|^2) - \frac{1}{6}(\|\delta_{\bar{x}} e^1\|^2 + \|\delta_{\bar{x}} e^0\|^2)\\
&\le \|\delta_t^+ e^0\|^2 + \frac{\gamma}{2}(\|B\delta_x^2 e^1\|^2 + \|B\delta_x^2 e^0\|^2) + \frac{2}{3}(\|\delta_x^+ e^1\|^2 + \|\delta_x^+ e^0\|^2)\\
&\le 2\|w\|^2.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
A^n &\le \exp(4c_8 T)(A^0 + \tau\sum_{n=1}^{N}\|q^n\|^2) \le \exp(4c_8 T)(2\|w\|^2 + \tau\sum_{n=1}^{N}\|q^n\|^2)\\
&\le \exp(4c_8 T)(2c_3^2 L(h^4+\tau^2)^2 + c_4^2 T L(h^4+\tau^2)^2).
\end{aligned}$$

Consequently,

$$\|\delta_x^+ e^n\| \le \sqrt{\exp(4c_8 T)(2c_3^2 L + c_4^2 T L)}(h^4+\tau^2). \tag{4.19}$$

Applying Lemma 3.3 gives that

$$\|e^n\| \le \frac{L\sqrt{6\exp(4c_8 T)(2c_3^2 L + c_4^2 T L)}}{6}(h^4+\tau^2). \tag{4.20}$$

Using Lemma 4.1 again, we have

$$\|e^n\|_\infty \le (M_1\sqrt{\exp(4c_8 T)(2c_3^2 L + c_4^2 T L)} + M_2\frac{L\sqrt{6\exp(4c_8 T)(2c_3^2 L + c_4^2 T L)}}{6})(h^4+\tau^2). \tag{4.21}$$

This completes the proof. □

5. Numerical results

In this section, we use the linear difference scheme (3.5)-(3.7) to calculate some numerical examples to verify the theoretical results given in previous sections. Denote the errors in the discrete maximum norm as

$$E(h,\tau) = \max_{0\le n\le N}\max_{1\le i\le M-1}|U_i^n - u_i^n|,$$

where U^n and u^n represent the exact solution (or the reference solution when the analytical solution is unknown) and numerical solution at time t_n calculated by h and τ, respectively. The convergence orders in space and time are defined by

$$\mathrm{Ord}_1 = \log_2(\frac{E(h,\tau)}{E(h,\tau/2)}),\ \mathrm{Ord}_2 = \log_2(\frac{E(h,\tau)}{E(h/2,\tau)}),$$

with respect to τ and h small enough, respectively.

Example 5.1. We first consider the following non-homogeneous strain wave equation with exact solution:

$$\begin{aligned}
&u_{tt} + 0.15\Delta^2 u - \Delta u + u^3 = g(x,t), x\in\Omega = [0,4], 0 < t \le 1,\\
&u(x,0) = \varphi(x), u_t(x,0) = \psi(x), x\in[0,4],
\end{aligned}$$

where $g(x,t) = (1+\pi^2+0.15\pi^4)\sin(\pi x)\exp(-t) + \sin^3(\pi x)\exp(-3t)$ and with initial conditions $\varphi(x) = -\psi(x) = \sin(\pi x)$. The exact solution of systems is given by $u(x,t) = \sin(\pi x)\exp(-t)$.

Table 1
The maximum errors and convergence orders obtained by the scheme (3.5)-(3.7) in Example 5.1.

τ	$h=1/200$		cpu	h	$\tau=1/6000$		cpu
	$E(h,\tau)$	Ord_1			$E(h,\tau)$	Ord_2	
1/5	2.6200e-2	—	2.6250 s	1/4	2.9000e-3	—	0.7656 s
1/10	7.3000e-3	1.8508	5.7813 s	1/8	1.8271e-4	3.9798	8.8594 s
1/20	1.9000e-3	1.9623	11.3438 s	1/16	1.1448e-5	3.9964	16.9844 s
1/40	4.7001e-4	1.9882	21.5313 s	1/32	**6.9621e-7**	4.0394	75.5781 s

Table 2
The maximum errors and convergence orders obtained by Scheme I in Example 5.1.

τ	$h=1/200$		cpu	h	$\tau=1/6000$		cpu
	$E(h,\tau)$	Ord_1			$E(h,\tau)$	Ord_2	
1/5	2.6500e-2	—	0.5469 s	1/4	1.2250e-1	—	0.5156 s
1/10	7.3000e-3	1.8521	0.9219 s	1/8	3.0100e-2	2.0269	1.0489 s
1/20	1.9000e-3	1.9880	2.3125 s	1/16	7.5000e-3	2.0069	1.5625 s
1/40	4.3185e-4	2.0998	5.1406 s	1/32	**1.9000e-3**	2.0017	31.6719 s

Table 3
The maximum errors and convergence orders in Example 5.2 with $T=1$.

τ	$h=1/256$		h	$\tau=1/640$	
	$E(h,\tau)$	Ord_1		$E(h,\tau)$	Ord_2
1/10	8.8300e-2	—	1/4	4.1000e-3	—
1/20	2.3100e-3	1.9348	1/8	3.0095e-4	3.7602
1/40	5.9000e-3	1.9646	1/16	1.9029e-5	3.9833
1/80	1.6000e-3	1.9328	1/32	1.2164e-6	3.9675

Table 4
E^n at $t=t_n$ for Example 5.2.

t	E^n
1	8.06551757173246
10	8.06551757173288
100	8.06551757173353
1000	8.06551757173432
10000	8.06551757173229

In Table 1 and Table 2, by fixing h and τ small enough respectively, we calculate the error and convergence order in space and time. From these data in Table 1, we clearly observe that the convergence order in the spatial direction is close to 4 while the convergence order in the time direction is close to 2, which is consistent with the previous theoretical analysis. Table 2 are calculated by using the linear scheme (denote Scheme I) constructed by T. Kadri [2]. By comparison, we can see that our algorithm is far superior to the Scheme I in accuracy and error perspective, and this phenomenon becomes more competitive as h decreases. But Scheme I is superior to our algorithm in calculation speed because of one need to calculate one more linear equation caused by auxiliary variable. It is worth noting that since the source term $g(x,t)$ is not equal to zero, the above discrete conservation laws are no longer valid, so we do not verify them here.

Example 5.2. We consider the homogeneous strain wave equation as follows

$$u_{tt}+0.05\Delta^2 u-\Delta u+u^3=0, x\in\Omega=[0,2], 0<t\le T,$$

$$u(x,0)=\sin(\pi x), u_t(x,0)=-\sin(\pi x), x\in[0,2].$$

Since the analytical solution is unknown, in order to verify the errors and convergence orders, we choose the numerical solution calculated by $h=1/256$ and $\tau=1/640$ as the reference solution. In Table 3, some numerical results similar to Example 5.1 are listed, and some similar phenomenon can be observed. Which shows that our method is also effective in this case.

In Table 4, we list the value of discrete energy E^n at time $t=t_n$ obtained by taking $\tau=1/100$ and $h=1/10$. We can clearly see that the difference schemes (3.5)-(3.7) (TSAV) can well maintain discrete energy. Fig. 1 shows that the evolution of discrete energy E^n of TSAV scheme and Scheme I obtained by taking $h=1/10$ and $\tau=1/200$ in a long time interval ($T=100$). It can be observed that TSAV scheme and Scheme I are good for conserving energy, but the former has higher accuracy.

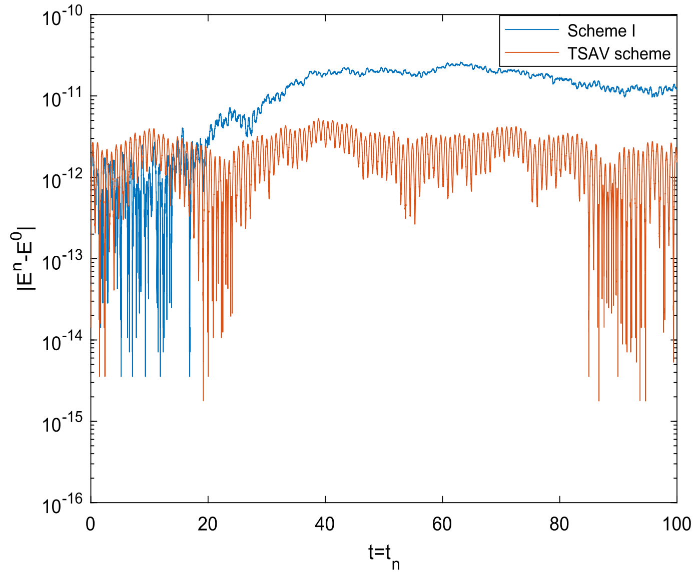

Fig. 1. The evolution of discrete energy E^n in a long time simulation.

6. Conclusion

In this paper, we proposed a high-order energy-preserving difference scheme for the initial boundary value problem of nonlinear fourth-order strain wave equation based on T-SAV approach. The proposed is proved to be linear, energy-preserving and convergent with $\mathcal{O}(h^4+\tau^2)$ in maximum error norm. Numerical experiments verify the effectiveness and accuracy of the difference scheme.

Data availability

Data will be made available on request.

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References

[1] J. Shen, Y. Yang, S. Chen, R. Xu, Finite time blow up of fourth-order wave equations with nonlinear strain and source terms at high energy level, Int. J. Math. 24 (2013) 1350043.
[2] T. Kadri, On the L^∞-convergence of two conservative finite difference schemes for fourth-order nonlinear strain wave equations, Comput. Appl. Math. 40 (2021) 1–31.
[3] R. Glassey, J.N. Schaeffer, Convergence of a second-order scheme for semilinear hyperbolic equation in 2 + 1 dimensions, Math. Comput. 56 (1991) 87–106.
[4] W. Strauss, L. Vazquez, Numerical solution of a nonlinear Klein-Gordon equation, J. Comput. Phys. 28 (1978) 271–278.
[5] L. An, A. Peirce, A weakly nonlinear analysis of elasto-plastic microstructure models, SIAM J. Appl. Math. 55 (1995) 136–155.
[6] J.A. Esquivel-Avila, Dynamics around the ground state of a nonlinear evolution equation, Nonlinear Anal., Theory Methods Appl. 63 (2005) e331–e343.
[7] Y. Liu, R. Xu, Fourth order wave equations with nonlinear strain and source terms, J. Math. Anal. Appl. 331 (2007) 585–607.
[8] T. Achouri, Conservative finite difference scheme for the nonlinear fourth-order wave equation, Appl. Math. Comput. 359 (2019) 121–131.
[9] J. Yang, J. Kim, The stabilized-trigonometric scalar auxiliary variable approach for gradient flows and its efficient schemes, J. Eng. Math. 129 (2021) 1–26.
[10] J. Shen, J. Xu, J. Yang, The scalar auxiliary variable (SAV) approach for gradient flows, J. Comput. Phys. 353 (2018) 407–416.
[11] D. Hou, M. Azaiez, C. Xu, A variant of scalar auxiliary variable approaches for gradient flows, J. Comput. Phys. 395 (2019) 307–332.
[12] F. Huang, J. Shen, Z. Yang, A highly efficient and accurate new scalar auxiliary variable approach for gradient flows, SIAM J. Sci. Comput. 42 (2020) A2514–A2536.
[13] Y. Fu, D. Hu, Y. Wang, High-order structure-preserving algorithms for the multi-dimensional fractional nonlinear Schrödinger equation based on the SAV approach, Math. Comput. Simul. 185 (2021) 238–255.
[14] Y. Fu, W. Cai, Y. Wang, A structure-preserving algorithm for the fractional nonlinear Schrödinger equation based on the SAV approach, arXiv preprint, arXiv:1911.07379, 2019.
[15] X. Hu, L. Zhang, A compact finite difference scheme for the fourth-order fractional diffusion-wave system, Comput. Phys. Commun. 182 (2011) 1645–1650.
[16] M. Ran, C. Zhang, New compact difference scheme for solving the fourth-order time fractional sub-diffusion equation of the distributed order, Appl. Numer. Math. 129 (2018) 58–70.
[17] S. Labidi, K. Omrani, A new conservative fourth-order accurate difference scheme for the nonlinear Schrödinger equation with wave operator, Appl. Numer. Math. 173 (2022) 0168.
[18] G. Zhang, Two conservative and linearly-implicit compact difference schemes for the nonlinear fourth-order wave equation, Appl. Math. Comput. 401 (2021) 126055.
[19] Z.Z. Shun, Numerical Methods of Partial Differential Equations, 2nd edn., Science Press, Beijing, 2012.
[20] A. Rouatbi, T. Achouri, K. Omrani, High-order conservative difference scheme for a model of nonlinear dispersive equations, Comput. Appl. Math. 37 (2018) 4169–4195.
[21] B.Y. Guo, P.J. Pascual, M.J. Rodriguez, L. Vázquez, Numerical solution of the sine-Gorden equation, Appl. Math. Comput. 18 (1986) 1–14.
[22] Y. Zhou, Application of Discrete Functional Analysis to the Finite Difference Methods, International Academic Publishers, Beijing, 1990.